Quantum Theory of an Optical Maser III.
Theory of Photoelectron Counting Statistics*

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ABSTRACT

In this paper we determine the photoelectron counting statistics produced by the fully quantum mechanical laser considered in the first paper of this series.[1] The problem of obtaining the photocount distribution from the now known photon statistics is derived in a completely quantum mechanical fashion. The time evolution of the combined photodetector-laser system is derived. The techniques developed for the solution of this problem are of general interest in the area of non-equilibrium quantum statistical dynamics.

I. Introduction

The photoelectron counting statistics produced by laser radiation has been the object of recent experimental[2-10] and theoretical[11-16] investigations. In this paper we analyze the problem in light of the quantum theory of the laser recently formulated by the authors.[1] The expression for the photon distribution of a laser operating in a steady state whether above, at, or below threshold was found to be

$$\rho_{n,n} = Z^{-1}\left\{ \frac{(A^2/BC)^{(n+A/B)}}{(n+A/B)!} \right\} \tag{1}$$

The photon statistical distribution is inferred in practice by photoelectron counting techniques. An object of the present paper is to obtain the distribution of photoelectrons produced by a fully quantum mechanical laser.

There are a number of admirable theoretical treatments of the photoelectron problem in the literature. However, for various reasons, we have developed the theory from first principles in this paper. Some of these reasons are listed below:

(1) The photocounting distribution is usually given as

$$P_m = \int dI\ P(I)\left[(\eta I)^m/m!\right] e^{-\eta I} \tag{2}$$

where

P_m = probability of m photocounts

I = intensity of light beam

η = a "quantum efficiency" of the counter

$P(I)$ = probability distribution for the light beam having the intensity I.

Equation (2) is not a convenient formulation of the photocounting distribution for our purpose as we most easily find the photon statistical distribution in the n representation. We obtain P(I) only after an auxiliary calculation of some complexity. The photoelectron distribution developed in the present paper has a simple physical interpretation.

(2) The usual procedure for obtaining the photocount distribution consists of counting the number of photoelectrons liberated from a photocathode in a time T, storing this information, and repeating this procedure a large number of times. The process of counting is a highly macroscopic affair. Due to the fact that this process is so complicated and microscopically disruptive,[17] it seems reasonable to assume that each count is equivalent to "looking" at the system. As is well known, upon looking at a system we destroy its wave function, i. e., when we trace over the laser coordinates, we produce a statistical mixture. Hence, one might well ask, "Does the procedure of looking after every count give the same counting distribution as would be observed if the system were not interrupted until a large number of potential counts had accumulated. We wish to investigate whether interruption of the combined detector-laser system involved in counting will give the same counting distribution as that obtained if the system were not interrupted until the time when all liberated electrons are counted. This

theoretical problem of leaving the combined laser-detector system undisturbed until time T does not correspond to the familiar experimental situation. However, it may be solved quantum mechanically and it is of interest to see how closely the results correspond to those obtained in the usual analysis.

(3) In the usual treatment of the problem the "quantum efficiency" η is given by

$$\eta = NCT$$

where

N = number of atoms in the photodetector

C = a parameter characterizing the detector depending on squared matrix elements, etc.

T = time of interaction between the photodetector and the laser.

Clearly this value of η results from a low order perturbation theory. To determine what happens for larger times T or numbers of photodetector atoms, we must extend the treatment to higher orders in η. In the present problem we find

$$\eta = 1 - e^{-NCT}$$

which is well behaved in the limit $NCT \to \infty$.

(4) Finally, the problem has an intrinsic interest in its own right in the field of non-equilibrium quantum statistical mechanics. That is, we must solve the problem of a quantized laser field interacting with many atoms. Clearly simple perturbation theory is not appropriate for this situation.

We develop a differential equation describing the laser-detector system which is then solved to obtain the time dependence of the laser-detector system. This approach is equivalent to summing an infinite set of diagrams.

In Section II we present our model and outline the method of calculation. Section III develops the analysis of this model. The photoelectron distribution implied by our quantum theory of a laser is found in Section IV. A concluding discussion is given in V.

II Model of a Photodetector

We take the photodetector to consist of N independent, equivalent but distinguishable atoms[18] each of which has a ground state $|g\rangle$ and a continuum of excited states $|k\rangle$, see Fig. 1. The atoms are placed in the cavity at time $t = 0$ in their ground state, while the radiation field is initially described by the diagonal density matrix $\rho_{n,n}(0)$. The combined laser-detector system will be described at later times by the density matrix.

$$\rho_{n,D;n,D'}(t)$$

where D and D' denote the states of the detector. For example, initially, the only nonvanishing elements of the density matrix are those given by

$$D = g(1),\ g(2), \ldots g(N)$$

while at some later time elements of the density matrix might exist for

$$D = k(1), k(2), g(3) \ldots k(N) \quad .$$

After a time T, the N atoms are removed from the cavity and the number of ionized atoms is determined (equal, of course, to the number of photoelectrons). This process is repeated several times in order to obtain the relative probability $P_m(T)$ for observing m photoelectrons irrespective of the state of the laser field. This is

$$P_m(T) = \sum_n P_{n,m}(T) \tag{3}$$

where $P_{n,m}(T)$ is the probability of finding the field in the state $|n\rangle$ with m photoelectrons ejected.

We now proceed with the calculation of $P_{n,m}(T)$. Since in our model, the probability of exciting any given configuration of m atoms is equal to the probability of exciting any other group of m atoms, it is clear that we need consider only the probability of exciting a specified group of m atoms (such as the first m). It is convenient to denote such a specific excited state by $D(m,\{k\})$ where m indicates that the first m atoms are excited and $\{k\}$ a state of the photodetector, for example

$$\{k\} = k(1)\cdots k(m),\ g(m+1)\cdots g(N) \ . \tag{4}$$

where each of the one electron states $k(i)$, $i = 1\cdots m$, is allowed to range over the continous spectrum of the i^{th} atom. In order to obtain the total probability of these atoms being ionized we must sum over the probabilities of finding each atom in any of its continuum states, that is

$$\mathcal{P}_{n,m}(T) \equiv P_{n,K(1)\cdots K(m),g(m+1)\cdots g(N)}(T)$$

$$= \sum_{\{k\}} \rho_{n,D(m,\{k\});n,D(m,\{k\})}(T) \qquad (5)$$

where the subscript K(m) on P means that the m^{th} atom is in any of its excited states. Explicitly this sum is

$$\sum_{k(1)} \cdots \sum_{k(m)} \rho_{n,k(1)\cdots k(m)\cdots g(N);n,k(1)\cdots k(m)\cdots g(N)}(T)$$

Once the probability $\mathcal{P}_{n,m}(T)$ of having the first m atoms ionized (while the field is in state $|n\rangle$), is known, the corresponding probability of any m atoms being excited is obtained by multiplying by the combinatorial factor $\binom{N}{m}$ representing the number of ways m atoms may be chosen from N, i.e.,

$$P_{n,m}(T) = \binom{N}{m}\mathcal{P}_{n,m}(T) \qquad (6)$$

We now proceed to obtain a differential equation for $P_{n,m}(T)$. The interaction Hamiltonian $\hbar V(t)$ for the combined laser-detector system is (in the interaction picture)

$$V(t) = -e \sum_{i=1}^{N} x_i(t)\ E(t) \equiv \sum_i V_i(t) \qquad (7a)$$

where e is the electronic charge and, E(t) is the quantized electric field operator[18a]

$$E(t) = e^{+i\nu a^+ at}\,\mathcal{E}(a+a^+)\,e^{-i\nu a^+ at} \qquad (7b)$$

while $x_i(t)$ is the position operator for the i^{th} atom,

$$x_i(t) = \exp\{i H_{atom} t\} x_i \exp\{-i H_{atom} t\} \tag{7c}$$

We will consider the only nonvanishing matrix elements of (7a) to be those connecting the ground state with an arbitrary excited state, and ~~we will~~ restrict ourselves to energy conserving transitions (work in the rotating wave approximation). The relevant matrix elements of (7a) are

$$\langle k(i), n | V_i(t) | g(i), n+1 \rangle = g\sqrt{n+1} \exp\{i(\omega_{k(i)} - \nu)t\} \tag{7d}$$

and

$$\langle g, n | V_i(t) | k, n-1 \rangle = g\sqrt{n} \exp\{-i(\omega_{k(i)} - \nu)t\} \tag{7e}$$

where the atomic frequency $\omega_{k(i)}$ is

$$\omega_{k(i)} = (\varepsilon_{k(i)} - \varepsilon_g)/h \quad , \tag{8}$$

and the coupling constant g is given by

$$g = -e\, x_{k,m} \mathcal{E}$$

It will be useful to denote the positive (7d) and negative frequency (7e) parts of $V_i(t)$ by $V(t)^{(-)}$ and $V(t)^{(+)}$ respectively so that

$$V(t) = \sum_i V_i(t)^{(+)} + V_i(t)^{(-)} \tag{9}$$

III Analysis

In order to obtain a differential equation for $P_{n,m}$ as used in Equation (3), we consider the equation of motion for the density matrix $\rho(t)$ of the detector-laser system (in the interaction picture

$$\dot{\rho}(t) = -i[V(t), \rho(t)] \tag{10}$$

The solution of Eq. (10) is formally

$$\rho(t) - \rho(0) = -i \int_0^t dt' [V(t'), \rho(t')] \tag{11}$$

For reasons which will become apparent it is more convenient to substitute Eq. (11) into Eq. (10) to obtain the alternative form[19] for the time rate of change of $\rho(t)$

$$\dot{\rho}(t) = (-i)^2 \, [V(t), \int_0^t dt' \, [V(t'), \rho(t')]] \tag{12}$$

Taking diagonal matrix elements of Eq. (12) for states $|n, D(m, \{k\})\rangle$ and summing over all excited states, we obtain an equation for $\rho_{n,m}(t)$

$$d\rho_{n,m}(t)/dt =$$

$$- \sum_{\{k\}} \langle n, D(m, \{k\}) | [V(t), \int_0^t dt' [V(t'), \rho(t')]] | n, D(m, \{k\}) \rangle \tag{13}$$

After a rather involved calculation, assuming only that the number of excited states is large, Eq. (13) reduces to

$$d\rho_{n,m}(t)/dt = -2r \, (N-m)n \, \rho_{n,m}(t) + 2r \, m(n+1) \, \rho_{n+1,m-1}(t) \tag{14}$$

where r is the transition rate given by Eq. (A.13a).

A detailed account of the transition from Eq. (13) to (14) is given in Appendix A.

Equation (14) refers to the probability of exciting the first m atoms. We proceed to find an equation for any m atoms by multiplying (14) by $\binom{N}{m}$ as implied by Eq. (6)

$$\dot{P}_{n,m} = -2Nr\,(N-m)n\,P_{n,m} + 2r\binom{N}{m}\,m(n+1)\wp_{n+1,m-1} \tag{15}$$

and since

$$m\binom{N}{m} = [N-(m-1)]\binom{N}{m-1}$$

Eq. (15) becomes

$$\dot{P}_{n,m} = -2\,r\,(N-m)n\;P_{n,m}(t) + 2\,r\,(N-m+1)(n+1)\;P_{n+1,m-1}(t) \tag{16}$$

Since normally $N >> m$ we write (16) as

$$\dot{P}_{n,m} = -\gamma n P_{n,m}(t) + \gamma(n+1)P_{n+1,m-1}(t) \tag{17}$$

where

$$\gamma = 2rN \tag{18}$$

It is clear from the form of Eq.'s (17) that this system of equations breaks up into an infinite set of equations, as depicted in Fig. 3, one set for each value of n_o, where

$$n_o = n+m \tag{19}$$

This is in accord with the obvious physical fact that the appearance of a photoelectron is associated with the disappearance of a photon. We denote the solution of one of these sets by

$$P_{n,m}(n_o,t) = P_{n_o-m,m}(n_o,t) \equiv P_m(n_o,t) \quad . \tag{20}$$

For any one of these sets, characterized by n_o, Eq. (17) becomes

$$\dot{P}_m(n_o,t) = -\gamma(n_o-m)P_m(n_o,t) + \gamma(n_o-(m-1))P_{m-1}(n_o,t) \tag{21}$$

Subject to the condition that initially

$$P_m(n_o,0) = \rho_{n_o,n_o} \ \rho_{g(1)\ldots g(N),g(1)\ldots g(N)} = \rho_{n_o,n_o} \tag{22}$$

the solution to Eq. (21) is

$$P_m(n_o,t) = \binom{n_o}{m} e^{-\gamma(n_o-m)t} (1-e^{-\gamma t})^m \ \rho_{n_o,n_o}(0) \tag{23}$$

as may be verified by direct differentiation. Eq. (23) may be written as

$$P_m(n_o,t) = \binom{n_o}{m} \ \eta^m(1-\eta)^{n_o-m} \ \rho_{n_o,n_o}(0) \tag{24}$$

where the "quantum efficiency" η is

$$\eta = (1-e^{-\gamma t}) \tag{25}$$

The solution of Eq. (17) is obtained from (24) by summing over $n_o = 1,2,3\ldots$, recognizing the constraint that $n+m=n_o$,

$$P_{n,m}(t) = \sum_{n_o} P_m(n_o,t)\,\delta_{n+m,n_o} \tag{26}$$

Finally we sum over all possible quantum states of the field (trace over the field) to obtain $P_m(t)$ as given in (3)

$$P_m(t) = \sum_n \sum_{n_o} \binom{n_o}{m} \eta^m (1-\eta)^{n_o-m} \rho_{n_o,n_o}(0)\,\delta_{n+m,n_o}$$

$$= \sum_{n_o} \binom{n_o}{m} \eta^m (1-\eta)^{n_o-m} \rho_{n_o,n_o}(0) \qquad (27)$$

which is the probability of detecting m photoelectrons if the quantum efficiency is η and the incident photon distribution is given by ρ_{n_o,n_o}.

IV. Photostatistics Implied by a Fully Quantized Laser

We may now calculate the photo-count distribution for a fully quantized laser by inserting $\rho_{n,n}$ as given by Eq. (1) into (25). The probability for finding m photoelectrons is

$$P_m = \sum_n \binom{n}{m} \eta^m (1-\eta)^{n-m} \left\{ Z^{-1} \frac{(A^2/BC)^{n+A/B}}{(n+A/B)!} \right\} \qquad (28)$$

Summing the series we find the basic relation

$$P_m = Z^{-1} \eta^m \frac{(A/BC)^{m+A/B}}{(m+A/B)!} \; {}_1F_1(m+1, m+\frac{A}{B}+1, (1-\eta)\frac{A^2}{BC}) \qquad (29)$$

where ${}_1F_1$ is the confluent hypergeometric function. Summarizing the notation

A = linear gain

B = nonlinear parameter

C = ν/Q

η = detector parameter

Z^{-1} = normalization factor which is given by

$$Z = \sum_n \frac{(A^2/BC)^{n+A/B}}{(n+A/B)!}$$

$$= \left[\frac{(A^2/BC)}{(A/B)!}\right] {}_1F_1(1;1+\frac{A}{B};\frac{A^2}{BC})$$

V. Discussion

Equation (27) may be understood as follows: Consider a state of the field having just one photon $|1\rangle$. Let the probability of having a photoelectron ejected from a detector interacting with this field for a certain time be given by η. Now if the state of the radiation field is $|n\rangle$ the probability of observing m photoelectrons should be proportional to

$$P_m^{(n)} \propto \eta^m \qquad (30)$$

which is to be multiplied by the probability that n-m quanta were not absorbed, i.e., $(1-\eta)^{n-m}$,

$$P_m^{(n)} \propto \eta^m (1-\eta)^{n-m} \qquad (31)$$

but we, of course, do not know which m photons of the original number n were absorbed so we must include a combinational factor

$$P_m^{(n)} = \binom{n}{m} \eta^m (1-\eta)^{n-m} . \qquad (32)$$

This is Bernoulli's distribution for m successful events (counts) and n-m failures, each event having a probability η. Now if we have a distribution of η values we must multiply (32) by $\rho_{n,n}$ and sum on n ,

$$P_m = \sum_n P_m^{(n)} \rho_{n,n}$$

$$= \sum_n \binom{n}{m} \eta^m (1-\eta)^{n-m} \rho_{n,n} \tag{33}$$

which is the result of (27) .

As a direct consequence of our model, (27) contains not only the small η limit ($\eta << 1$), but is valid for all η ($0 \leq \eta \leq 1$). Clearly, if we wish to obtain the photon statistics by counting photoelectrons we must require $\eta = 1$ for then, as we see from Eq. (27),

$$P_m = \rho_{m,m} \quad .$$

In all other cases ($\eta < 1$) we are measuring the photoelectron statistics which in general will be very different, e. g. , Eq. (29) for the laser is

$$P_m = \rho_{m,m} \left\{ \eta^m \, {}_1F_1 \left(m+1, m+\frac{A}{B} + 1, (1-\eta) \frac{A^2}{BC} \right) \right\}$$

To cast (33) into another form we write ρ in the $P(\alpha)$ representation[20]

$$\rho_{n,n} = \int d^2\alpha \, P(\alpha) \left[(\alpha^*\alpha)^n / \, n! \right] \exp \left\{ -\alpha^*\alpha \right\} \tag{34}$$

so that (33) becomes

$$P_m = \int d^2\alpha \sum_{n=m} \binom{n}{m} P(\alpha) \left[(\alpha^*\alpha)^n / n!\right] \exp\{-\alpha^*\alpha\} \, \eta^m \, (1-\eta)^{n-m}$$

let n = ℓ+m

$$= \int d^2\alpha \exp\{-\alpha^*\alpha\} \left[(\alpha^*\alpha\,\eta)^m / m!\right] P(\alpha) \sum_{\ell=0} \left[(\alpha^*\alpha)^\ell / \ell!\right] (1-\eta)^\ell$$

$$= \int d^2\alpha \; P(\alpha) \left[(\alpha^*\alpha\,\eta)^m / m!\right] \exp\{-\alpha^*\alpha\eta\} \tag{35}$$

When $\alpha^*\alpha$ is associated with the field intensity Eq. (35) is recognized as being the same as (2).

Finally we note that the usual procedure for obtaining Eq. (2) breaks the time T into many small time intervals. In each of these small time intervals a quantum mechanical calculation is carried out in low order perturbation theory to obtain the probability of obtaining a count. The calculation is then completed by classical probabitistic arguements for the number of counts observed in the larger time interval T. In this type of analysis one is "looking" at the system at the end of each of the small time intervals. Our result (27) or (35) agrees with that obtained in the usual discussion.

It should perhaps be pointed out that if the number of photoatoms is not much larger than the number of ejected photoelectrons then we should use Eq. (16) instead of (17). For example, if we only have 10 atoms we could never observe more than 10 photoelectrons. Further, we have neglected the fact that atoms deeper inside the photodetector would see a weaker field. It might also be noted that the presence of the photodetector would slightly load the laser in a way which could be described by giving the cavity a smaller Q value. These points could be included in the theory but are of secondary interest.

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Appendix

This appendix justifies Eq. (14) beginning from (13)

$$d\rho_{n,m}(t)/dt =$$

$$-\sum_{\{k\}} \langle n, D(m,\{k\}) | [V(t), \int_0^t dt' [V(t'), \rho(t')]] | n, D(m,\{k\}) \rangle \tag{13}$$

Inserting V(t) as given by (9) we have

$$d\rho_{n,m}(t)/dt = -\int_0^t dt' \sum_{k(i)} \sum_{\ell(j)} \sum_{i,j}$$

$$\times \langle n, k(i), \ell(j) | [(V_i(t)^{(+)} + V_i(t)^{(-)}), [(V_j(t')^{(+)} + V_j(t')^{(-)}), \sum_{\{k\}}{}' \langle \mathcal{D} | \rho(t') | \mathcal{D} \rangle]] | n, k(i), \ell(j) \rangle \tag{A.1}$$

where the primed sum means no sum over k(i) and ℓ (j) and $|\mathcal{D}\rangle$ denotes the set of detector states for N-2 atoms omitting the i^{th} and j^{th} atoms. It is convenient to introduce the notation

$$\sum_{\{k\}}{}' \langle \mathcal{D} | \rho(t) | \mathcal{D} \rangle = R(t) \tag{A.2}$$

It will be shown that only those terms in (A.1) with i = j and k = ℓ contribute significantly. Consider a typical term in (A.1) with $i \neq j$.

$$\sum_{i \neq j} \sum_{k(i)} \sum_{\ell(j)} \langle n, k(i), \ell(j) | V_i^{(-)} \exp - i(\omega_{k(i)} - \nu)t$$

$$\times V_j^{(+)} \exp i(\omega_{\ell(j)} - \nu)t'$$

$$\times R(t') | k(i), \ell(j), n \rangle \tag{A.3}$$

Writing Eq. (A. 3) in matrix element form we find

$$\int dt' \sum_{i\neq j} \sum_{k(i)} \sum_{\ell(j)} \Big\{ \langle n, k(i), g(j) | V_i^{(-)} | n-1, g(i), g(j) \rangle \exp -i(\omega_{k(i)} - \nu)t$$

$$\times \langle n-1, g(i), g(j) | V_j^{(+)} | n, g(i), \ell(j) \rangle \exp i(\omega_{\ell(j)} - \nu)t'$$

$$\times \langle n, g(i), \ell(j) | R(t') | n, k(i), g(j) \rangle \qquad \text{(A. 4)}$$

We proceed by considering the density of excited states to be so large that we may replace the sum over excited states by integrations over atomic frequencies with the appropriate density of states $\sigma(\omega)$. Expression (A. 4) then becomes

$$\int_0^t dt' \sum_{i\neq j} \int d\omega_i \, \sigma(\omega_i) \int d\omega_j \, \sigma(\omega_j)$$

$$\times \langle n, k(i) | V_i^{(-)} | n-1, g(i) \rangle \exp -i(\omega_{k(i)} - \nu)t$$

$$\times \langle n-1, g(j) | V_j^{(+)} | n, \ell(j) \rangle \exp i(\omega_{\ell(j)} - \nu)t' \qquad \text{(A. 5)}$$

$$\times \langle n, g(i) \ell(j) | R(t') | n, k(i), g(j) \rangle$$

Now in general all the factors except the exponential appearing in (A. 5) will be slowly varying functions of atomic frequency, so we take them outside the integral sign and perform the integration over ω_i and ω_j. When these integrations are carried out we obtain a delta function in time from each integration

$$\int d\omega_i \ \exp -i(\omega_i - \nu)t \rightarrow \delta(t)$$

$$\int d\omega_j \ \exp i(\omega_j - \nu)t' \rightarrow \delta(t')$$

Hence, when we integrate over t' we find that Equation (13) is proportional to the elements of the density matrix which are off diagonal in the atomic states, and vanish at time t=0 as the system is prepared with all the "photoatoms" in their ground states.

$$\langle n, g(i), \ell(j) \,|\, R(0) \,|\, n, k(i), g(j) \rangle = \langle n, g(i), \ell(j) \,|\, \rho(0) \,|\, n, k(i), g(j) \rangle = 0 \tag{A. 6}$$

Similarly, terms with j=i will vanish, unless $k(i)=\ell(i)$ i.e., in Equation (A. 1) we need keep only the terms with i=j and $k=\ell$. Equation (A. 1) now reads

$$\begin{aligned} d\mathcal{P}_{n,m}(t)/dt &= (-i)^2 \sum_i \sum_{k(i)} \int_0^t dt' \\ &\times \langle n, D(m, \{k\}) \,|\Big\{ \Big(V_i(t)^{(+)} + V_i(t)^{(-)}\Big)\Big(V_i(t')^{(+)} + V_i(t')^{(-)}\Big) \rho(t') \\ &+ \rho(t') \Big(V_i(t')^{(+)} + V_i(t')^{(-)}\Big)\Big(V_i(t)^{(+)} + V_i(t)^{(-)}\Big) \\ &- \Big(V_i(t)^{(+)} + V_i(t)^{(-)}\Big) \rho(t') \Big(V_i(t')^{(+)} + V_i(t')^{(-)}\Big) \\ &- \Big(V_i(t')^{(+)} + V_i(t')^{(-)}\Big) \rho(t') \Big(V_i(t)^{(+)} + V_i(t)^{(-)}\Big)\Big\} \,|\, n, D(m, \{k\}) \rangle \end{aligned} \tag{A. 7}$$

In view of the relations 8a and 8b the only nonvanishing terms in Eq. (A. 1) are

$$d\mathcal{P}_{n,m}(t)/dt = \sum_{k(1)} \cdots \sum_{k(m)} (-i)^2 \int_0^t dt'$$

$$\times \langle n, D(m,\{k\}) | \Bigg\{ \sum_{i=1}^{m} \sum_{k(i)} \left(V_i(t)^{(-)} V_i(t')^{(+)} \rho(t') \right) + \sum_{i=m+1}^{N} \sum_{k(i)} \left(V_i(t)^{(+)} V_i(t')^{(-)} \rho(t') \right)$$

$$+ \sum_{i=1}^{m} \sum_{k(i)} \left(\rho(t') V_i(t)^{(-)} V_i(t')^{(+)} \right) + \sum_{i=m+1}^{N} \sum_{k(i)} \left(\rho(t') V_i(t')^{(+)} V_i(t)^{(-)} \right) \qquad \text{(A. 8)}$$

$$- \sum_{i=1}^{m} \sum_{k(i)} \left(V_i(t)^{(-)} \rho(t') V_i(t')^{(+)} \right) - \sum_{i=m+1}^{N} \sum_{k(i)} \left(V_i(t)^{(+)} \rho(t') V_i(t')^{(-)} \right)$$

$$- \sum_{i=1}^{m} \sum_{k(i)} \left(V_i(t')^{(-)} \rho(t') V_i(t)^{(+)} \right) - \sum_{i=m+1}^{N} \sum_{k(i)} \left(V_i(t')^{(+)} \rho(t') V_i(t)^{(-)} \right) | n, D(m,\{k\}) \rangle$$

As we are from now on only interested in diagonal elements of ρ let us introduce the notation

$$\langle n, D | \rho | n, D \rangle \equiv \rho(n, D) \qquad \text{(A. 9)}$$

Certain terms in Eq. (A. 1) may be neglected. These terms are those involving integrals over the excited state frequencies such as

$$\int d\omega_i \exp\{i(\omega_i - \nu)(t-t')\} \rho(t, k(i), n) \qquad \text{(A. 10)}$$

where the density matrix corresponds to the excited state k(i). In this case we note that since there are a large number of states η to which the electron ma[y] be excited $\rho(t', k(i), n)$ is to a good approximation

$$\rho(t', k(i), n) \cong \sum_{k(i)} \frac{\rho(t', k(i), n)}{\eta} \equiv \frac{\rho(t', K(i), n)}{\eta} \qquad \text{(A. 11)}$$

which vanishes as η becomes very large. The arguement K(i) in (A. 11) means that we have summed over the excited states of the i^{th} atom. Finally the dominate terms in (A. 1) are

$$d\mathcal{P}_{n,m}(t)/dt = -\int_0^t dt' \sum_{i=m+1}^{N} \sum_{k(i)} \langle n, g(i) | V_i(t)^{(+)} V_i(t')^{(-)} | n, g(i) \rangle \rho(n, K(1) .. K(m) .. g(N), t' \ldots - c.c.$$

$$+ \int_0^t dt' \sum_{i=1}^{m} \sum_{k(i)} \langle n, k(i) | V_i(t)^{(-)} V_i(t')^{(+)} | n, k(i) \rangle \rho(n+1), K(1) .. g(i) .. K(m) .. g(N) \ldots + c.c. \qquad \text{(A, 12)}$$

where as in (A. 11) the symbol K(m), for example, denotes the fact that the m^{th} atom is in any of its excited states. If we replace the sum on k(i) as it appears in (A. 12) by an integral over ω_i

$$\sum_{k(i)} \rightarrow \int d\omega_i \, \sigma(\omega_i)$$

and note that $\rho(t', D(m, \{k\}), n)$ does not depend on k(i) (i. e. the i^{th} atom is in its ground state since $i > m$) the integral over ω_i leads to a delta function, $\delta(t-t')$. This delta function when accompanied by the integral on t'

$$d\wp_{n,m}(t)/dt = -\sum_{i=m+1}^{N} 2rn\ \rho\big(n, K(1)\ldots K(m)\ldots g(N), t\big) \qquad (A.13).$$

$$+\sum_{i=1}^{m} 2r(n+1)\ \rho\big((n+1), K(1)\ldots g(i)\ldots K(m)\ldots g(N), t\big)$$

where $$r = \pi\left|x_{k,i}\right|^2 \sigma(\nu) \qquad (A.13a)$$

Recalling that the probability of exciting any given atom is the same as we any other we write

$$\rho\big(n, K(1)\ldots g(i)\ldots K(m), g(m+1)\ldots g(N)\big) = \wp_{n+1,n-1} \qquad (A.14)$$

and noting that the first sum in (A.13) (m+1 to N) leads to a factor N-m, while the second sum (1 to m) is replaced by m (A.13) becomes

$$d\wp_{n,m}(t)/dt$$

$$= -\ 2r(N-m)\ n\wp_{n,m}(t) + 2r\ m(n+1)\wp_{n+1,m-1}(t)$$

Footnotes

1. M. Scully and W. E. Lamb, Jr., Phys. Rev. 159, 208 (1967). Symbols used in Eq. (1) are summarized in section 4 of the present paper.
2. A Summary of experimental results is given in Proceedings of the International Conference on the Physics of Quantum Electronics, Puerto Rico 1965, edited by P. Kelley, B. Lax, and P. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1965). See especially: J. A. Armstrong and A. W. Smith, ibid., p. 701; F. Johnson, T. McLean and E. Pike, ibid., p. 706; C. Freed and H. A. Haus, ibid., p. 715.
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16. B. R. Mollow, to be published.

17. When the elections are liberated they are accelerated by an external electric field and "banged" into a photosensitive surface.

18. The photodetecting medium will be taken to be optically thin so that each atom sees the same radiation field as any other atom.

18a. In the notation of Ref. I, ν is the frequence of the optical radiation, a^+ and a are the creation and annihilation operators, and $\mathcal{E}$ is a constant having the units of an electric field.

19. A contribution involving $\rho(0)$ is omitted in (12) as it does not give any matrix elements of interest to us.

Figure Captions

1. Pictorial representation of photodetector consisting of N independent atoms. Each atom in detector has a ground state $|g\rangle$ and continum of excited states $|k\rangle$. Atoms are labeled by indexing atomic state with particle number eg. $|k(m)\rangle$.

2. Figure indicating the grouping of Eq.'s (17) according to value of n_o. First example is for $n_o = 5$. Dashed line is for arbitrary n_o.